A Linear Algorithm for Resource Tripartitioning
Triconnected Planar Graphs

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Abstract. Given a connected graph \( G = (V, E) \), a set \( V_r \subseteq V \) of \( r \) special vertices, three distinct base vertices \( u_1, u_2, u_3 \in V \) and three natural numbers \( r_1, r_2, r_3 \) such that \( r_1 + r_2 + r_3 = r \), we wish to find a partition \( V_1, V_2, V_3 \) of \( V \) such that \( V_i \) contains \( u_i \) and \( r_i \) vertices from \( V_r \), and \( V_i \) induces a connected subgraph of \( G \) for each \( i, 1 \leq i \leq 3 \). We call a vertex in \( V_r \) a resource vertex and the problem above of partitioning vertices of \( G \) as the resource 3-partitioning problem. In this paper, we give a linear-time algorithm for finding a resource tripartition of a 3-connected planar graph \( G \). Our algorithm is based on a nonseparating ear decomposition of \( G \) and \( st \)-numbering of \( G \). We also present a linear algorithm to find a nonseparating ear decomposition of a 3-connected planar graph. This algorithm has bounds on ear-length and number of ears.

Keywords: Algorithm, Nonseparating ear decomposition, Planar graph, Resource tripartitioning, Resource bipartitioning, \( st \)-numbering, Triconnected graph.

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1 Introduction

Let \( G = (V, E) \) be a connected graph of \( |V| = n \) vertices. Among these \( n \) vertices of \( G \), some belong to a special class of vertices that we call resource vertices. Let \( V_r \subseteq V \) be the set of resource vertices and \( |V_r| = r \). Let \( u_1, u_2, u_3 \in V \) be three designated vertices and \( r_1, r_2, r_3 \) be three natural numbers such that \( r_1 + r_2 + r_3 = r \). Our goal is to find a partition \( V_1, V_2, V_3 \) of \( V \) such that \( u_1 \in V_1, u_2 \in V_2, u_3 \in V_3 \), \( V_i \) contains \( r_i \) resource vertices and \( V_i \) induces a connected subgraph of \( G \) for each \( i, 1 \leq i \leq 3 \). We call this partitioning of vertices a resource 3-partitioning of \( G \). For example, Figure 1(a) shows a connected graph \( G \) with \( n = 15, r = 8 \) vertices, where each resource vertex is drawn by white circle. Figure 1(b) illustrates a resource 3-partition of \( G \) for \( r_1 = 3, r_2 = 2, r_3 = 3 \).

The resource tripartitioning problem is a special case of...
of resource \(k\)-partitioning problem, for \(k = 3\). A resource \(k\)-partitioning is defined as partition \(V_1, V_2, \ldots, V_k\) of \(V\) with a set \(V_r \subseteq V\) of \(r\) resource vertices, base vertices \(u_1, u_2, \ldots, u_k \in V, k\) natural numbers \(r_1, r_2, \ldots, r_k\) such that \(\sum_{i=1}^{k} r_i = r\), where \(u_i \in V_i, V_i\) contains \(r_i\) resource vertices and \(V_i\) induces a connected subgraph of \(G\) for each \(i, 1 \leq i \leq k\).

Resource partitioning has significant applications in various areas. In computer networks, we may consider printers, routers, scanners etc. as resources. Resources need to be partitioned to balance loads on these resources and to prevent network traffic bottleneck. Furthermore, in multimedia networks, it is desired to assign a server to a specific group of clients for balancing loads among the servers. Again, in electrical power distribution systems, resource partitioning has another real-time application to serve consumers better. Here, distributed resources include a variety of energy sources like turbines, photovoltaics, fuel-cells and storage devices with various capacities. Distribution of these resources among the demand centers offers increased reliability, lower cost of power delivery and additional supply flexibility.

Resource partitioning has its application in the fault-tolerant routing of communication networks [13, 22] and in computational aspects, too. For example, in grid computing, we wish to divide a complex task such as computation of fractals into several subtasks and then we wish to delegate each of these subtasks to a computing element in the grid such that a computing element in the grid might not be overwhelmed with tasks from various other clients. This concept is applicable to telecommunication networks, fault tolerant systems, various producer-consumer problems and so on.

A related problem is a \(k\)-partitioning problem in which we are given a graph \(G = (V, E), k\) distinct base vertices \(u_1, u_2, \ldots, u_k \in V\), and \(k\) natural numbers \(n_1, n_2, \ldots, n_k\) such that \(\sum_{i=1}^{k} n_i = |V|\), we wish to find a partition \(V_1, V_2, \ldots, V_k\) of the vertex set \(V\) such that \(u_i \in V_i; |V_i| = n_i; V_i\) induces a connected subgraph of \(G\) for each \(i, 1 \leq i \leq k\).

The \(k\)-partitioning problem is \(NP\)-complete in general [7]. Although not every graph has a \(k\)-partition, Gyoőri and Lovász independently proved that every \(k\)-connected graph has a \(k\)-partition for any \(u_1, u_2, \ldots, u_k\) and \(n_1, n_2, \ldots, n_k\) [9, 14]. However, their proofs do not yield any polynomial time algorithm for actually finding a \(k\)-partition of a \(k\)-connected graph. For the case \(k = 2, 3, 4\), following algorithms have been known:

(i) There is a linear-time algorithm to find a bipartition of a biconnected graph [19, 20].

(ii) There is an \(O(n^2)\) time algorithm to find a 3-partition of a triconnected graph [20].

(iii) There is a linear-time algorithm to find a 4-partition of a four connected planar graph with base vertices located on the same face of the given graph [17].

On the other hand, polynomial-time algorithms have not been known for the case \(k \geq 4\). A polynomial-time algorithm for any \(k\) is claimed in [15], but is not correct [17]. If all the vertices are resource vertices then resource \(k\)-partitioning and \(k\)-partitioning problem are the same. Thus resource \(k\)-partitioning problem is also \(NP\)-complete. [18] claims the resource \(k\)-partitioning problem to be \(NP\)-hard but their claim is not correct. The following algorithms are known for finding a resource \(k\)-partition of a graph for \(k = 2, 3, 4\).

(i) There are linear-time algorithms to find resource bipartitions of path-reducible graphs, series-parallel graphs and connected graphs where all resource vertices are contained in the same biconnected component [18].

(ii) There is an \(O(n^2)\) time algorithm to find vertex-subset tripartitions (equivalent to resource tripartitions [18]) of triconnected and 3-edge-connected graphs [21].

(iii) There is an \(O(n)\) algorithm to find a resource four-partition of a 4-connected planar graph with four base vertices located on the same face of a planar embedding.

But there exists no polynomial-time algorithm for resource \(k\)-partitioning of graphs for \(k > 3\). In this paper, we give a linear algorithm for finding a resource tripartition of a 3-connected planar graph based on a “nonseparating ear decomposition” of the given graph. “Nonseparating ear decomposition” is a generalization of “canonical decomposition” [1]. “Canonical decomposition” is applied in convex grid drawing of planar graph [5]. “Canonical decomposition” i.e. “canonical ordering” has applications in producing straight line grid drawings with polynomial sizes for planar graphs. A “canonical decomposition”, a “realizer”, a “Schnyder labeling” and an “orderly spanning tree” of a plane graph play an important role in straight-line drawings, floorplanning, graph encoding etc. [2, 4, 6, 10, 12]. Miura et. al. proved that a “canonical decomposition”, a “realizer”, a “Schnyder labeling”, an “orderly spanning tree” and an “outer triangular convex grid drawing” are notions equivalent with each other [16]. Hence

\footnote{1T. Awal, Resource partitioning on planar graphs, M.Sc. Engineering Thesis, Department of Computer Science and Engineering, Bangladesh University of Engineering and Technology, Dhaka, November 2007.}
for any $a,b,c$ we present a linear-time algorithm for finding a resource tripartitioning of a 3-connected planar graph $G$. In section 2 we define several graph theoretical terms used in this paper.

In this section we define several graph theoretical terms used in this paper.

Let $G(V,E)$ be a connected simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $n$ the number of vertices in $G$ and by $m$ the number of edges in $G$. Thus $n = |V(G)|$, $m = |E(G)|$. An edge joining vertices $u,v$ is denoted by $(u,v)$. The degree of a vertex $v$ in a graph $G$, denoted by $d(v)$, is the number of edges incident to $v$ in $G$. The connectivity $κ(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph $K_1$. We say that $G$ is $k$-connected if $κ(G) ≥ k$. We call a vertex of $G$ a cut vertex if its removal results in a disconnected graph or a single-vertex graph. A walk, $v_0, e_1, v_1, \ldots, v_l$, in a graph $G$ is a sequence of vertices and edges of $G$, beginning and ending with a vertex, in which each edge is incident to two vertices immediately preceding and following it. If the vertices $v_0, v_1, \ldots, v_l$ are distinct (except possibly $v_0, v_l$), then the walk is called a path and usually denoted either by the sequence of vertices $v_0, v_1, \ldots, v_l$ or by the sequence of edges $e_1, e_2, \ldots, e_l$. The length of a path is $l$ which is one less than the number of vertices on the path. A path or walk is open if $v_0 ≠ v_l$. A path or walk is closed if $v_0 = v_l$. A closed path containing at least one edge is called a cycle. For a path $P$, $V_{in}(P)$ denotes the internal vertices of $P$, i.e., all the vertices except the endpoints. For $W ⊆ V$, we denote by $G−W$ the graph obtained from $G$ by deleting all vertices in $W$ and all edges incident to them.

Let $s$ and $t$ be any two vertices of a connected graph $G$. An $st$-numbering of $G$ is a numbering of its vertices by integers $1,2,\ldots,n$ such that a vertex $s$ receives number 1, a vertex $t$ receives number $n$ and every other vertex of $G$ is adjacent to at least one lower-numbered vertex and at least one higher-numbered vertex. An interesting property of $st$-numbering of a graph is shown in the following fact.

(st1) If a graph $G$ has an $st$-numbering $π = v_1,v_2,\ldots,v_n$, then both the subgraphs of $G$ induced by $\{v_1, v_2, \ldots, v_i\}$ and $\{v_i+1, v_i+2, \ldots, v_n\}$ are connected for each $i, 1 ≤ i ≤ n$.

Not every connected graph has an $st$-numbering but the following Lemma [8] holds.

**Lemma 2.1** Let $G$ be a biconnected undirected graph and $(s,t)$ be any edge of $G$. Then $G$ has an $st$-numbering $π = v_1,v_2,\ldots,v_n$ such that $v_1 = s$ and $v_n = t$, and $π$ can be found in linear time.

An ear decomposition of a biconnected graph $G$ is a decomposition $G = P_0∪P_1∪\ldots∪P_k$, where $P_0$ is a path or cycle and $P_i$, $0 ≤ i ≤ k − 1$, is a path with only its two distinct end vertices in common with $P_k ∪ P_{k−1}∪\ldots∪P_{i+1}$. An ear is a path. An open ear is a path with two distinct end vertices. We call an ear a trivial ear if the length of the ear is one. We call an ear a non-trivial ear if the length of the ear is greater than one. Let $a,b,c$ be three vertices in $G$ and $P_i$ be open paths on $G$. We denote by $G_i$ the subgraph of $G$ induced by the edges of $P_0 ∪ P_1∪\ldots∪P_i$, by $G_{k}$ the subgraph of $G$ induced by the edges of $P_{i+1}∪P_{i+2}∪\ldots∪P_{k}$. So $G_k = G$. Then $P_0, P_1,\ldots,P_k$ is a nonseparating ear decomposition (nsed) through vertices $a,b$ and避免 vertex $c$ if the following five conditions hold.

(nsed1) If $(a,b) ∈ E(G)$, then $P_k$ is a cycle containing the edge $(a,b)$. Otherwise, $P_k$ is a path in $G$ with the vertices $a,b$ as endpoints.

(nsed2) The first non-trivial ear has only one internal vertex and the internal vertex is $c$.

(nsed3) For each $i, 0 ≤ i < k$, $P_i$ is a path connecting 2 distinct vertices of $G_i$ and $V_{in}(P_i) ∩ V(G_i) = \phi$.

(nsed4) For each $i, 0 ≤ i ≤ k$, $G_i$ is connected.

(nsed5) For each $i, 1 ≤ i ≤ k$, each internal vertex of $P_i$ has a neighbor in $G_{i−1}$.

Let augmented graph, $G_i^* = (V(G_i), E(G_i)∪\{(a,b)\})$ and augmented $P_k^* = (V(P_k), E(P_k) ∪ \{(a,b)\})$. For both of $P_k^*$ and $G_i^*$, $(a,b) ∉ E(G)$ and $(a,b)$ becomes an outer edge of $G_i$. Note that $P_k^*$ is a cycle and...
formed by the edges on the boundary of the face. We denote the contour of the outer face of graph through \( k \).

Figure 2(a) shows a planar graph with a fixed embedding. A plane graph \( G \) is a graph that is drawn by thick lines. A plane graph \( G \) is planar if it can be embedded in the plane such that no two edges intersect geometrically except at a vertex to which they are both incident. A plane graph divides the plane into connected regions called faces. We regard the contour of a face as a clockwise cycle formed by the edges on the boundary of the face. We denote the contour of the outer face of graph \( G \) by \( C_o(G) \).

We write \( C_o(G) = w_1, w_2, \ldots, w_h, w_1 \) if the vertices \( w_1, w_2, \ldots, w_h \) on \( C_o(G) \) appear clockwise in this order, as illustrated in Figure 3. We call a vertex an outer vertex and an edge an outer edge, if the vertex and edge respectively lie on \( C_o(G) \). We call a path \( P \) in a biconnected plane graph \( G \) a chord-path of \( G \) if \( P \) satisfies the following conditions.

(i) \( P \) connects two outer vertices \( w_p, w_q, p < q \);
(ii) \( \{w_p, w_q\} \) is a separation pair of \( G \);
(iii) \( P \) lies on an inner face; and
(iv) \( P \) does not pass through any outer edge and any outer vertex other than the ends \( w_p \) and \( w_q \).

Figure 3: A plane graph with chord-paths \( P_1, P_2, P_3, P_4 \)

The plane graph \( G \) in Figure 3 has four chord-paths \( P_1, P_2, \ldots, P_4 \) drawn by thick lines. A chord-path \( P \) is minimal if none of \( w_{p+1}, w_{p+2}, \ldots, w_{q-1} \) is an end of a chord-path. Thus the definition of a minimal chord-path depends on which vertex is considered as the starting vertex \( w_1 \) of \( C_o(G) \). \( P_1 \) and \( P_3 \) in Figure 3 are minimal, while \( P_2 \) and \( P_4 \) are not minimal. Let \( \{v_1, v_2, \ldots, v_p\} \), \( p \geq 3 \), be a set of three or more outer vertices consecutive on \( C_o(G) \) such that \( d(v_1) \geq 3 \), \( d(v_2) = d(v_3) = \ldots = d(v_{p-1}) = 2 \), and \( d(v_p) \geq 3 \). Then we call the set \( \{v_2, v_3, \ldots, v_{p-1}\} \) an outer chain of \( G \). The graph in Figure 3 has two outer chains \( \{w_4, w_5\} \) and \( \{w_8\} \). We call an outer chain \( \{v_2, v_3, \ldots, v_{p-1}\} \) of \( G \) a good outer chain if the outer chain does not contain any vertex of \( V(P_k) \). An outer chain \( \{v_2, v_3, \ldots, v_{p-1}\} \) of \( G \) is a bad outer chain if the outer chain contains a vertex of \( V(P_k) \). We call an outer edge \( (u, w) \) of \( G \) a good outer edge if \( (u, w) \notin E(P_k) \), \( d(u) \geq 3 \) and \( d(w) \geq 3 \) in \( G \), and \( u \in V(G_i) \) or \( w \in V(G_j) \).

We say that a plane graph \( G \) is internally triconnected if \( G \) is biconnected and, for any separation pair \( \{u, v\} \) of \( G \), \( u \) and \( v \) are outer vertices and each connected component of \( G - \{u, v\} \) contains an outer vertex. In other words, \( G \) is internally 3-connected if and only if it can be extended to a 3-connected graph by adding a vertex in an outer face and connecting it to all outer vertices. If a biconnected plane graph \( G \) is not internally 3-connected, then \( G \) has a separation pair \( \{u, v\} \) of one of the three types illustrated in Figure 4. If an internally 3-connected plane graph \( G \) is not 3-connected, then \( G \) has a separation pair of outer vertices and hence \( G \) has a chord-path when \( G \) is not a single cycle.

Figure 4: Biconnected graphs which are not internally 3-connected

A resource bipartitioning of a connected graph \( G \) of \( |V| = n \) vertices can be defined as follows. Let \( V_r \subseteq V \) be the set of resource vertices and \( |V_r| = r \). Let \( u_1, u_2 \in V \) be two designated vertices and \( r_1, r_2 \) be two natural numbers such that \( r_1 + r_2 = r \). Our goal is to find a partition \( V_1, V_2 \) of \( V \) such that \( u_1 \in V_1, u_2 \in V_2, V_1 \) contains \( r_1 \) resource vertices and \( V_2 \) induces a connected subgraph of \( G \) for each \( i, 1 \leq i \leq 2 \). The resource bipartitioning problem is a special case of the resource \( k \)-partitioning problem, for \( k = 2 \). We have the following lemmas on resource bipartitioning.

**Lemma 2.2** A resource bipartition of a biconnected graph \( G \) can be found in linear time [21, 18].

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Lemma 2.3 Let \( s, t \) be the two base vertices in a connected graph \( G \). If \( G \cup \{(s, t)\} \) is biconnected, then a resource bipartition of \( G \) can be found in linear time.

Proof. We can find a resource bipartition of the biconnected graph \( G \cup \{(s, t)\} \) in linear time by Lemma 2.2. As the vertices \((s, t)\) would belong to two different partitions, clearly a resource bipartition of \( G \cup \{(s, t)\} \) is a resource bipartition of \( G \). Q.E.D.

In section 3, we provide the constructive proof of the existence of a nonseparating ear decomposition through two vertices \( a, b \) and avoiding a third vertex \( c \) of a tri-connected planar graph \( G \) for any \( a, b, c \).

3 Nonseparating Ear Decomposition

In this section, we show a constructive proof for the existence of a nonseparating ear decomposition through two vertices \( a, b \) and avoiding a third vertex \( c \) of a tri-connected planar graph \( G \) for any \( a, b, c \).

Lemma 3.1 Let \( u_1, u_2, \ldots, u_l \) be the outer facial vertices of an internally 3-connected planar graph \( G \) and \( S = \{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\} \) be an outer chain of \( G \), then \( G - S \) is internally triconnected.

Proof. Let \( u_p, u_q \) be the two ends of a minimal chord-path \( P \) of \( G \) such that \( p < q \). Assume for the sake of contradiction that \( G - S \) is not internally triconnected. Then \( G - S \) has either a cut-vertex or a separation pair \( \{u, v\} \). We first consider the case where \( G - S \) has a cut-vertex \( v \). Then \( v \) must be an outer vertex of \( G - S \) and \( v \neq u_p, u_q \). Otherwise, \( G \) would not be internally triconnected. Then the minimal chord-path \( P \) must pass through \( v \), as illustrated in Figure 5, contrary to the definition of a chord path.

![Figure 5: P passes through an outer vertex in \( G_i \)](image)

We now consider the case where \( G - S \) has a separation pair \( \{u, v\} \) having one of the three types shown in Figure 4. Then \( \{u, v\} \) would be a separation pair of \( G \) having one of the three types illustrated in Figure 4. Hence \( G \) would not be internally triconnected, a contradiction. Q.E.D.

Theorem 3.2 Let \( a, b, c \) be any three vertices in a tri-connected planar graph \( G \). Then \( G \) has a nonseparating ear decomposition through \( a, b \) and avoiding \( c \) and it can be found in linear time.

Proof. Without loss of generality we may assume that \( c \) is on the outer boundary of \( G \). If \((a, b) \in E(G)\), then let \( P_k \) be an inner facial cycle passing through \((a, b)\) and not passing through \( c \). If \((a, b) \notin E(G)\) and both of \( a, b \) are outer vertices of \( G \), then let \( P_k \) be the outer facial path from \( a \) to \( b \) not passing through \( c \). If \((a, b) \notin E(G)\) and at least one of \( a, b \) is an inner vertex of \( G \), then let \( P_k \) be the path from \( a \) to \( b \) that does not contain any outer edge and go through \( c \). Since \( G \) is 3-connected, \( P_k \) exists in all of the three cases mentioned above and (nsed1) holds for \( P_k \).

Let \( e_1, e_2, \ldots, e_l \) be the neighbours of \( c \) where \( l = d(c) \). We set \( P_0, P_1, \ldots, P_{l-2} \) as \((e, e_1), (e, e_2), \ldots, (e, e_{l-2})\) respectively. We set \( P_{l-2} = (e_{l-1}, e, e_l) \). \( P_{l-2} \) is the first non-trivial ear and it has exactly one internal vertex which is \( c \). Hence (nsed2) holds. Since \( P_0, P_1, \ldots, P_{l-2} \) are open ears and \( V_m(P_{l-2}) \cap V(G_{l-2}) = \varnothing \), (nsed3) holds for \( P_0, 0 \leq j \leq l - 2 \). As for each \( j, 0 \leq j \leq l - 2 \), \( G_j \) is connected, (nsed4) holds for \( P_j, 0 \leq j \leq l - 2 \). As \( P_0, P_1, \ldots, P_{l-3} \) are trivial ears and the internal vertex \( c \) of \( P_{l-2} \) has at least a neighbour \( e_1 \) in \( G_{l-3} \), (nsed5) holds for \( P_j, 0 \leq j \leq l - 2 \). Clearly \( G_j \) or \( G_j^* \) is internally triconnected for each \( j, 0 \leq j \leq l - 2 \).

Assume for inductive hypothesis that the ears \( P_0, P_1, \ldots, P_i \) for \( l - 2 \leq i \leq k - 2 \) have been chosen appropriately so that (nsed2) holds for the first non-trivial ear \( P_{i-2} \) and (nsed3), (nsed4), (nsed5) hold for each index \( j, 0 \leq j \leq i \). Furthermore \( G_i^* \) or \( G_i \) is internally triconnected for each index \( j, 0 \leq j \leq i \). We now show that there is an ear \( P_{i+1} \) in \( G_i \) such that (nsed3), (nsed4), (nsed5) hold and \( G_i^* \) or \( G_i \) is internally triconnected for the index \( j = i + 1 \). Let \( u_1, u_2, \ldots, u_l \) be the outer facial vertices of \( G_i \) or \( G_i^* \). We have the following cases to consider.

Case 1: \( G_i \) or \( G_i^* \) is 3-connected.

We only consider the case where \( G_i \) is 3-connected, since the proof for the case where \( G_i^* \) is 3-connected is similar. Since \( G_i \) is 3-connected, every vertex of \( G_i \) has degree at least three. There is at least a vertex \( u \) on \( C_0(G_i) \) such that \( u \in V(G_i) \) and \( u \) has a neighbour \( w \) on \( C_0(G_i) \) such that \((u, w) \notin E(P_k)\). Otherwise, at least one of \( a, b \) would be an inner vertex of \( G \) and \( P_k \) would contain an outer edge of \( G \) (see Figure 6(a)) or both of \( a, b \) would be outer vertices of \( G \) and \( P_k \) would contain an inner edge of \( G \) (see Figure 6(b)), a contradiction. Then \((u, w) \) is a good outer edge of \( G_i \), as illustrated in Figure 7. We set \( P_{i+1} = (u, w) \). As \((u, w) \) is a trivial open ear, (nsed3) holds for \( P_{i+1} \). As \((u, w) \) is a good outer edge of \( G_i \), \( G_{i+1} \) remains connected and hence (nsed4) holds for \( P_{i+1} \). As \((u, w) \) is a trivial ear, (nsed5) also holds for \( P_{i+1} \). Clearly \( G_{i+1} \) is internally...
We first consider the case where such that . As is not triconnected. Since has an outer cycle and hence there is a chord-path of for , we set . Clearly (nsed3), (nsed4), (nsed5) hold for , and remains internally triconnected.

(ii) has no outer chain.

Case 2: \((a, b) \notin E(G)\) and \(G_i\) or \(G_i^*\) is not 3-connected.

We first consider the case where \((a, b) \notin E(G)\) and \(G_i\) is not triconnected. Since \(i + 1 \leq k - 1\), \(G_i\) is not a single cycle and hence there is a chord-path of \(G_i\). Let be the two ends of a minimal chord-path \(P\) of \(G_i\) such that \(p < q\). As \(\{u_p, u_q\}\) is a separation pair of \(G_i\), then \(q \geq p + 2\). We have the following two subcases to consider.

Case 2a: \(P_k\) is not on \(C_o(G_i)\).

We have the following two subcases to consider.

(i) \(G_i\) has an outer chain \(\{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}\).

If \(\{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}\) is a good outer chain, we choose \(P_{i+1} = (u_p, u_{p+1}, u_{p+2}, \ldots, u_{q-1}, u_q)\). As \(u_p \neq u_q\) and \(V(G_{i+1}) = \emptyset\), (nsed3) holds for \(P_{i+1}\). As \(G\) is triconnected and each internal vertex of \(P_{i+1}\) has degree two in \(G_i\), each of the internal vertices of \(P_{i+1}\) has a neighbor in \(G_i\). So \(G_{i+1}\) is also connected. Thus (nsed4) and (nsed5) hold for \(P_{i+1}\). From \(G_{i+1}\) is internally triconnected.

We thus assume that \(\{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}\) is a bad outer chain of \(G_i\). In this case \(G_i - \{u_p, u_q\}\) has 2 components. There is at least a vertex \(v \in G_i - \{u_p, u_q\}\) in the component not containing \(u \in \{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}\) such that \(v \in V(G_i)\) and \(d(v) \geq 3\) in \(G_i\). Otherwise, \(G\) would not be triconnected or \(d(v) = 2\) in \(G_i\) and \(v\) would be contained in a good outer chain, a contradiction. The vertex \(v\) has a neighbor \(w \in V(G_i) - \{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}\) such that \(d(w) \geq 3\) in \(G_i\) and \((v, w) \notin E(P_k)\). Otherwise, \(d(w) = 2\) in \(G_i\) and \(w\) would be contained in a good outer chain, a contradiction. Then \((v, w)\) is a good outer edge of \(G_i\), as illustrated in Figure 8. We set \(P_{i+1} = (v, w)\). Clearly (nsed3), (nsed4), (nsed5) hold for \(P_{i+1}\), and \(G_{i+1}\) remains internally triconnected.

(ii) \(G_i\) has no outer chain.

In this case every vertex in \(\{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}\) has degree at least three in \(G_i\). Otherwise, \(G_i\) would have an outer chain. Furthermore, there is at least a vertex \(w \in C_o(G_i)\) such that \(d(w) \geq 3\) in \(G_i\) and \((u, w) \notin E(P_k)\), we set \(P_{i+1} = (u, w)\). Then \((u, w)\) is a good outer edge of \(G_i\) and (nsed3), (nsed4), (nsed5) hold for \(P_{i+1}\), and \(G_{i+1}\) remains internally triconnected. If \(w\) has no such neighbor \(w\), then there is at least a vertex \(v \in G_i - \{u_p, u_q\}\) in the component not containing \(w \in \{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}\) such that \(v \in V(G_i)\) and \(d(v) \geq 3\) in \(G_i\). Otherwise, \(G\) would not be triconnected or \(d(v) = 2\) in \(G_i\) and \(v\) would be contained in a good outer chain, a contradiction. The vertex \(v\) has a neighbor \(w \in V(G_i) - \{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}\) such that \(d(w) \geq 3\) in \(G_i\) and \((v, w) \notin E(P_k)\). Otherwise, \(d(w) = 2\) in \(G_i\) and \(w\) would be contained in a good outer chain, a contradiction. Then \((v, w)\) is a good outer edge of \(G_i\). We set \(P_{i+1} = (v, w)\). Clearly
(nsed3), (nsed4), (nsed5) hold for $P_{i+1}$, and $G_{i+1}$ remains internally triconnected.  
Case 2b: $P_k$ is on $C_o(G_i)$ or $C_o(G_i^*)$.  
If removal of $P_k$ has left $P_k$ on $C_o(G_i)$, we augment $G_i$ to $G_i^*$ as stated before. Otherwise, $P_k$ has already been on $C_o(G_i)$ and $G_i$ has already been augmented to $G_i^*$. Hence it is sufficient to consider only the case for $G_i^*$. We have the following subcases to consider.  
(i) $G_i^*$ has an outer chain $\{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}$.  
In this case $\{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}$ is a good outer chain. Otherwise, $P_k$ is not on $C_o(G_i)$. We choose $P_{i+1} = (u_p, u_{p+1}, u_{p+2}, \ldots, u_{q-1}, u_q)$. Clearly (nsed3), (nsed4), (nsed5) hold for $P_{i+1}$, and $G_{i+1}$ remains internally triconnected.  
(ii) $G_i^*$ has no outer chain.  
In this case every vertex in $\{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}$ has degree at least three in $G_i^*$. Otherwise, $G_i^*$ would have an outer chain. Furthermore, there is at least a vertex $u \in \{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}$ such that $u \in V(G_i^*)$. Otherwise, $\{u_p, u_q\}$ would be a separation pair of $G$ and $G$ would not be triconnected. Clearly $u$ has a neighbour $w \in \{u_{p+1}, u_{p+2}, \ldots, u_{q-1}\}$ such that $(u, w) \notin E(P_k)$. Then $(u, w)$ is a good outer edge of $G_i^*$. We set $P_{i+1} = (u, w)$. Clearly (nsed3), (nsed4), (nsed5) hold for $P_{i+1}$, and $G_{i+1}$ remains internally triconnected.  
Now it remains to consider the case where $(a, b) \notin E(G)$ and $G_i^*$ is not triconnected. In this case, we choose $P_{i+1}$ similar to the subcase (2b) with the exception that we do not need any augmentation.  
Case 3: $(a, b) \in E(G)$ and $G_i$ is not 3-connected.  
In this case we choose $P_{i+1}$ similar to the case where $(a, b) \notin E(G)$ and $G_i^*$ is not triconnected.  
Thus the existence of a nonseparating ear decomposition of $G$ through $a, b$ and avoiding $c$ for any $a, b, c$ is proven. An algorithm for finding a nonseparating ear decomposition based on the proof above can be implemented. We have to keep track of outer chains, minimal chord-paths and candidate degree three vertices of $G_i$. Each face is traversed at most a constant number of times. So run time is linear. Hence the Theorem 3.2 follows.  
Q.E.D.  

We call the algorithm obtained from this constructive proof for the existence of a nonseparating ear decomposition of $G$ through $a, b$ and avoiding $c$ for any $a, b, c$ Algorithm Find-Decomposition.  

Lemma 3.3 Let $a, b, c$ be three vertices in a triconnected planar graph $G$ and $n$, $m$ denote the number of vertices and the number of edges in $G$ respectively. Then the nonseparating ear decomposition of $G$ through $a, b$ and avoiding $c$ produced by Algorithm Find-Decomposition has the following properties.  

(a) if $(a, b) \in E(G)$, then  
(i) length of any ear is at least one and at most the length of the longest facial cycle of $G$.  
(ii) the number of ears is $m - n + 1$.  
(b) if $(a, b) \notin E(G)$, then  
(i) length of any ear is at least one and at most the larger one of the length of $P_k$ and one less than the length of the longest facial cycle of $G$.  
(ii) the number of ears is $m - n + 2$.  

Proof. (a)(i) From the constructive proof of Theorem 3.2, it can be found that each ear $P_i, i \neq k$, is either a single edge or a path with all of its internal vertices belonging to an outer chain of $G_{i-1}$. So the length of a non-trivial ear $P_i$ can be at most 1 less than that of the longest facial cycle. But as $(a, b) \in E(G)$, the ear $P_k$ is set as an inner facial cycle passing through the edge $(a, b)$ and not passing through $c$. Therefore, $P_k$ may have the length of the longest facial cycle. Hence length of any ear is at least one and at most the length of the longest facial cycle of $G$.  
(a)(ii) We employ an induction on $n$. A graph must have at least $n = 4$ vertices and $m = 6$ edges to be 3-connected. A nonseparating ear decomposition of the graph in Figure 10(i) is as follows. $P_0 = a, c, P_1 = d, c, b$ and $P_k = a, d, b, a$. So we have $3 = 6 - 4 + 1 = m - n + 1$ ears for any $a, b, c$ of $G$.  

Figure 10: (i) A triconnected graph with four vertices and six edges  
(ii) $G^\alpha$ with $n'$ vertices and $m'$ edges.  

Assume that $n \geq 5$ and the result is true for all triconnected planar graphs having $n$ vertices. Without loss of generality we may assume that $c$ is on the outer boundary of $G$. We now add a vertex $\alpha$ on the outer face of $G$. Let $G^\alpha = G \cup \{\alpha\}$. To make $G^\alpha$ 3-connected without losing planarity, $d(\alpha)$ number of edges are added from $\alpha$ to its $d(\alpha)$ number of neighbours on $C_o(G), \text{ where } 3 \leq d(\alpha) \leq |C_o(G)|$. $G^\alpha$ has $n' = n + 1$ vertices and $m' = m + d(\alpha)$ edges. To find a nonseparating ear decomposition of $G^\alpha$ (see Figure 10(ii) ) we use the decomposition as in $G$ until the vertex $\beta$ or $\gamma$ or $\theta$ is contained in $G_i$. Then the following ears are chosen in the decomposition of $G_i$.  

\begin{figure}[h]  
\centering  
\includegraphics[width=0.5\textwidth]{figure10.png}  
\caption{(i) A triconnected graph with four vertices and six edges  
(ii) $G^\alpha$ with $n'$ vertices and $m'$ edges.}  
\end{figure}
In this section we give an algorithm to find a resource 3-partition of a graph $G$ having a nonseparating ear decomposition through two vertices $a, b$ and avoiding a third vertex $c$ for any $a, b, c$.

### 4 Resource Tripartition

In this section we give an algorithm to find a resource 3-partition of a planar graph $G$ having a nonseparating ear decomposition through two vertices $a, b$ and avoiding a third vertex $c$. Then for any $W \subseteq V_{in}(P_i)$, $H_i - W$ is connected for $i, 0 \leq i \leq q$.

**Algorithm Resource_Tripartition**

**Input:** A planar graph $G = (V, E)$ which has a nonseparating ear decomposition through two vertices $a, b$ and avoiding a third vertex $c$ for any $a, b, c$, three designated distinct vertices $u_1, u_2, u_3$ and three natural numbers $r_1, r_2, r_3$ such that $\sum_{i=1}^{3} r_i = r$.

**Output:** A resource 3-partition of $G$.

**begin**

Find a nonseparating ear decomposition $P_0, P_1, \ldots, P_q$ of $G$ through $u_1, u_2$ and avoiding $u_3$ of $G$;

Let $P_0, P_1, \ldots, P_q$ be the non-trivial ears of this nonseparating ear decomposition of $G$;

Let $i$ be the minimum integer such that $V_{in}(P_0) \cup V_{in}(P_1) \cup \ldots \cup V_{in}(P_i)$ contains at least $r_3$ resource vertices, where each $P_i$ is a non-trivial ear, $0 \leq i \leq q$;

Let $e$ be the excess number of resource vertices in $V_{in}(P_0) \cup V_{in}(P_1) \cup \ldots \cup V_{in}(P_i)$ over $r_3$;

There are the following two cases: (1) $e = 0$, and (2) $e \geq 1$;

Case 1: $e = 0$.

In this case, $H_i$ contains $r_3$ resource vertices, and $H_i$ contains $r_1 + r_2$ resource vertices.]

Let $V_3 = H_i$.

Find a resource bipartition $V_1, V_2$ of the biconnected graph $H_i \cup \{(u_1, u_2)\}$ such that $u_1 \in V_1, u_2 \in V_2$, $V_1$ contains $r_1$ resource vertices and $V_2$ contains $r_2$ resource vertices, and both $V_1, V_2$ induce connected subgraphs;

We can find a resource bipartition of $H_i \cup \{(u_1, u_2)\}$ in linear time by Lemma 2.3

**return** $V_1, V_2, V_3$ as a resource 3-partition of $G$.

Case 2: $e \geq 1$.

In this case, $H_i$ contains $r_3 + e$ resource vertices, and $H_i = H_{i-1} - V_{in}(P_i)$ contains $r_1 + r_2 - e$ resource vertices. Since $e \geq 1$, $V_{in}(P_i)$ contains at least two resource vertices, $|V_{in}(P_i)| \geq 2$ and hence $V_{in}(P_1)$ is an outer chain of $H_{i-1}$.

Let $C_o(H_{i-1}) = w_1, w_2, \ldots, w_k, w_1$ where $w_1 = u_1$;

Assume that $V_{in}(P_i) = \{w_{p+1}, w_{p+2}, \ldots, w_q-1\}$ is an outer chain of $H_{i-1}$;

Find an $st$-numbering $v_1, v_2, \ldots, v_q$ of $H_i \cup \{(u_1, u_2)\}$ such that $s = v_1 = u_1$ and $t = v_q = u_2$;

Let $w_p = v_{p'},$ and $w_q = v_q'$;

Assume that $p' < q$, otherwise interchange the roles of $w_1$ and $w_2$;

Let $v_1, v_2, \ldots, v_{p'}$ contain $x$ resource vertices;

There are the following three subcases: (a) $r_1 \leq x$, (b) $x + e \leq r_1$, and (c) $x < r_1 < x + e$;

**Subcase 2a:** $r_1 \leq x$.

In this subcase, the last $e'$ vertices containing $e$ resource vertices in the outer chain $V_{in}(P_i)$ are added to $H_i$ as the deficient $e$ resource vertices.

Let $V_1 = \{v_1, v_2, \ldots, v_{n_1}\}$ be the first $n_1$ vertices containing $r_1$ resource vertices in the $st$-numbering of $H_i \cup \{(u_1, u_2)\}$;

Let $V'_2 = \{v_{n_1+1}, v_{n_1+2}, \ldots, v_{n_2}\}$ be the remaining vertices containing $r_2 - e$ resource vertices in $H_i$, where $w_q = v_q \in V'_2$;

By the fact (st1) of an $st$-numbering both $V_1$ and $V'_2$ induce connected graphs.

Let $W = \{w_{q-1}, w_{q-2}, \ldots, w_{q'-e'}\}$ be the set of the last $e'$ vertices containing $e$ resource vertices in $V_{in}(P_i)$;

Let $V_2 = V'_2 \cup W$;

Since $w_{q-1}$ is adjacent to $w_q \in V'_2, V_2$ induces a connected graph with $r_2$ resource vertices.


Let $V_3 = H_i - W$;
[ V_3 is connected by Lemma 4.1, and has $r_3$ resource vertices. ]

return $V_1,V_2,V_3$ as a resource 3-partition of $G$.

Subcase 2h: $x + e \leq r_1$.

[In this subcase, the first $e'$ vertices containing $e$ resource vertices in $V_{in}(P_i)$ are added to $H_i$ as the deficient $e$ resource vertices. ]

Let $V'_1 = \{v_1,v_2,\ldots,v_{n_1}\}$ be the first $n_1$ vertices containing $|V_1 - e$ resource vertices in the $st$-numbering of $H_i \cup \{(n_1,u_2)\}$, where $w_p = v'_p \in V'_1$;

Let $V_2 = \{v_{n_1+1},v_{n_1+2},\ldots,v_{n_1+e}\}$ be the remaining vertices containing $r_2$ resource vertices in $H_i$;

[ By the fact (st1) of an st-numbering both $V'_1$ and $V_2$ induce connected graphs. ]

Let $W = \{w_{p+1},w_{p+2},\ldots,w_{p+e}\}$ be the set of the first $e'$ vertices containing $e'$ resource vertices in $V_{in}(P_i)$;

Let $V_1 = V'_1 \cup W$;

[ Since $w_{p+1}$ is adjacent to $w_p \in V'_1$, $V_1$ induces a connected graph with $r_1$ resource vertices. ]

Let $V_3 = H_i - W$;

[ $V_3$ is connected by Lemma 4.1, and has $r_3$ resource vertices. ]

return $V_1,V_2,V_3$ as a resource 3-partition of $G$.

Subcase 2c: $x < r_1 < x + e$.

[ In this subcase, $e \geq 2$; the first $b$ vertices containing $r_1 - x$ resource vertices and the last $c$ vertices containing $e - (r_1 - x)$ resource vertices in $V_{in}(P_i)$ are added to $H_i$ as the deficient $e$ resource vertices. ]

Let $W = \{w_{p+1},w_{p+2},\ldots,w_{p+b}\}$ be the set of the first $b$ vertices containing $r_1 - x$ resource vertices in $V_{in}(P_i)$;

Let $W' = \{w_{q-1},w_{q-2},\ldots,w_{q-c}\}$ be the set of the last $c$ vertices containing $e - (r_1 - x)$ resource vertices in $V_{in}(P_i)$;

[ Since $|W| + |W'| = b + c < |V_{in}(P_i)|$, $W \cap W' = \emptyset$, $|W \cup W'| = b + c$ and $W \cup W'$ contains $e$ resource vertices. ]

Let $V_1 = \{v_1,v_2,\ldots,v_{p}\} \cup W$;

Let $V_2 = \{v_{p+1},v_{p+2},\ldots,v_{p+b}\} \cup W'$;

[ $V_1$ and $V_2$ contain $r_1$ and $r_2$ resource vertices respectively, $w_p = v'_p \in V_1$, $w_q = v'_q \in V_2$, and both $V_1$ and $V_2$ induce connected subgraphs. ]

Let $V_3 = H_i - W \cup W'$;

[ $V_3$ is connected by Lemma 4.1, and has $r_3$ resource vertices. ]

return $V_1,V_2,V_3$ as a resource 3-partition of $G$.

end.

Since st-numbering can be obtained in $O(n)$ time by Lemma 2.1, the running time of the above algorithm is $O(n)$ if a nonseparating ear decomposition of $G$ through $u_1,u_2$ and avoiding $u_3$ can be found in linear time. Thus we have the following theorem.

**Theorem 4.2** A planar graph $G$ having a nonseparating ear decomposition through two vertices $a,b$ and avoiding a third vertex $c$ for any $a,b,c$ has a resource 3-partition. Furthermore, if a nonseparating ear decomposition of $G$ through two vertices $a,b$ and avoiding a third vertex $c$ for any $a,b,c$ can be found in linear time, a resource 3-partition of $G$ can be found in linear time.

Hence from Theorem 3.2, we obtain a linear algorithm to find a resource 3-partition of a 3-connected planar graph $G$ by using the nonseparating ear decomposition of $G$ through $a,b$ and avoiding $c$ described in section 3.

5 Conclusion

In this paper, we present a linear-time algorithm for finding a resource tripartition of a planar graph for which a nonseparating ear decomposition through two vertices $a,b$ and avoiding a third vertex $c$ for any $a,b,c$ can be found in linear time. We also present a linear algorithm for constructing a nonseparating ear decomposition of a triconnected planar graph. The interesting features of the nonseparating ear decomposition produced by our algorithm regarding the number of ears and bounds on length of ear can have significant applications. Using our algorithm for finding a nonseparating ear decomposition, we obtain a linear algorithm to find resource tripartitions of triconnected planar graphs. Applying our algorithm for finding a nonseparating ear decomposition with an algorithm in [3], we can also achieve a linear algorithm to find three “independent spanning trees” in 3-connected planar graphs rooted at a vertex $r$. However, the following problems related to resource partitioning are still open.

(a) Developing algorithms for finding resource $k$-partitions of graphs for $k \geq 4$.

(b) Developing algorithms for finding resource $k$-partitions of graphs for $k \geq 2$ where resources are specified for the partitions.

Acknowledgement

Jou et. al. presented a linear algorithm for finding a nonseparating ear decomposition of a triconnected planar graph [11] but their description of the algorithm is ambiguous and does not have any proof. However, their work served as an inspiration to our research. We thank the anonymous referees for their useful comments.
References


